**Agenda**

- Renewal theory connection
  - Stationary transition probabilities
- Random 1D walk
  - Null chain example
- Discrete chains continued
  - Boundary transition rates
  - Random walks on graphs
Renewal Theory Connection

• In this lecture, we examine Markov chains for which stationary distribution $\pi$ exists
  - We will then show that it is the solution to the same equation $\pi = \pi P$

• Let $T_{ij}$ be the first passage time from state $i$ to state $j$:
  $$T_{ij} = \min(n \geq 1 : X_n = j | X_0 = i)$$

• For $i = j$, the visit is called a return
  - $T_{ii}$ is the random variable giving the number of steps (delay) before the chain returns to state $i$ after it started in state $i$
  - Notice that $T_{ii}$ does not depend on how the chain arrived into state $i$ (Markov property)
Renewal Theory Connection 2

• Suppose $f_{ij}$ is the probability that the chain ever visits state $j$ starting in state $i$

• **Definition**: state $j$ is *recurrent* if $f_{jj} = 1$ and *transient* otherwise (i.e., $f_{jj} < 1$)

• **Definition**: assume $j$ is recurrent; if $E[T_{jj}] < \infty$, then $j$ is called *positive*; otherwise, it is called *null*

• Define a renewal process $M_{ij}(n)$ to count number of visits to state $j$ by time $n$ assuming that $X_0 = i$

$$M_{ij}(n) = \sum_{k=1}^{n} I_{ij}(k)$$

$$I_{ij}(n) = \begin{cases} 
1 & X_n = j \\
0 & \text{otherwise}
\end{cases}$$
Renewal Theory Connection 3

- Graphical illustration:

\[ I_{ij}(n) \]

\[ W_1 \quad W_2 \quad W_3 \quad W_4 \]

- Note that \( W_1 \) has the same distribution as \( T_{ij} \)
  - All of the remaining \( W_k, k \geq 2 \), are iid random variables \( T_{jj} \), which yields from the Elementary Renewal Theorem

\[
\lim_{n \to \infty} \frac{E[M_{ij}(n)]}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} p_{ij}^{(k)} = \frac{1}{E[T_{jj}]}\]
• From this point on, assume a positive chain

• Recall that

\[ a^{(n)} = aP^n \]

- and

\[ \pi = \lim_{n \to \infty} \frac{1}{n} \sum_{s=1}^{n} a^{(s)} = \lim_{n \to \infty} \frac{a}{n} \sum_{s=1}^{n} P^s \]

• From the previous slide, this limit exists and thus:

\[ \pi_i = \frac{1}{E[T_{ii}]} \]

• Additionally, since

\[ \pi P = \lim_{n \to \infty} \frac{a}{n} \sum_{s=1}^{n} P^{s+1} = \pi \]

- we get

\[ \pi = \pi P \]
Random 1D Walk

• Example

\[ P = \begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots & \ldots \\ q & 0 & p & \ldots & \ldots & \ldots \\ 0 & q & 0 & p & \ldots & \ldots \\ 0 & 0 & q & 0 & p & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \]

• The chain represents a random walk with a reflective (non-absorbing) boundary at 0
  - At each step, it goes left with probability \( q \) and right with probability \( p \)
  - Except one special case when it bounces off zero with probability 1
Random 1D Walk 2

- We assume that \( p + q = 1 \) and \( pq > 0 \)

- Next, we analyze the stationary distribution of this infinite chain

- Write:
  \[
  \begin{align*}
  \pi_0 &= q\pi_1 \\
  \pi_1 &= \pi_0 + q\pi_2 \\
  \pi_j &= p\pi_{j-1} + q\pi_{j+1}
  \end{align*}
  \]

- Solving this recursively and using induction:
  \[
  \begin{align*}
  \pi_1 &= \pi_0/q \\
  \pi_2 &= (p/q)\pi_0/q \\
  \vdots & \quad \vdots \quad \vdots \quad \vdots \\
  \pi_j &= (p/q)^{j-1}\pi_0/q, \ j \geq 1
  \end{align*}
  \]
• Note, however, that we still do not know $\pi_0$
  - This is accomplished using normalization since all $\pi_j$ must sum up to 1:

$$1 = \sum_{j=0}^{\infty} \pi_j = \pi_0 \left( 1 + \frac{1}{q} \sum_{j=1}^{\infty} \left( \frac{p}{q} \right)^{j-1} \right)$$

  - When $p < q$, the sum is finite and all states are positive (the walk is “pulled” towards zero and $E[T_{jj}]$ is finite for all states $j$):

$$\pi_0 = \frac{q - p}{2q}$$

  - If $p = q = 1/2$, then all states are null (the expected duration before return to each state is infinity)

  - Finally, if $p > q$, then all states are transient, i.e., the chain keeps drifting towards infinity
**Example**

- A PhD student goes through 3 states
  - Find the probability that on a given day the student is weird

Note: matlab eig(A) produces right eigenvectors; to get the left ones, transpose A first, i.e., eig(A')
Transition Rates

• In what follows, we establish a useful rule that allows a simpler computation of $\pi$

• First, notice that the number of transitions into and out of a given state $j$ are almost the same
  - Define $A_j(n)$ to be the number of arrivals into $j$ by time $n$ and $D_j(n)$ the number of departures (including self-loops)
  - Clearly, the difference between these two metrics is no more than 1 at any time $n$

• Thus, arrival and departure rates are asymptotically the same:

$$\left| r_A - r_D \right| = \left| \frac{A_j(n)}{n} - \frac{D_j(n)}{n} \right| = \frac{|A_j(n) - D_j(n)|}{n} \to 0$$
**Transition Rates 2**

- Similarly observe the following
  - The probability to find a recurrent chain in state $j$ is equal to the rate of transition from all states (including $j$) into $j$

$$
\pi_j = \sum_{i=0}^{\infty} \pi_i p_{ij}
$$

- To prove this, notice that this is an expansion of $\pi_j$ from equation $\pi = \pi P$

- Consider the packet-loss chain (note: variables are different from last time to simplify formulas):

```
1 - p
  0  p
  q
  1
1 - q
```
Transition Rates 3

• For this example, we can write:

\[ \pi_0 = \pi_0(1 - p) + \pi_1 q \]

  - Or in other words:

\[ \frac{\pi_0}{\pi_1} = \frac{q}{p} \]

  - Since \( \pi_0 + \pi_1 = 1 \), we have

\[ \pi_0 = \frac{q}{p + q} \]

• Another way to look at rates is to compute the total transition rate out of and into each state:

\[ \pi_i \sum_{j \neq i} p_{ij} = \sum_{j \neq i} \pi_j p_{ji} \]
For the same example:

- The transition rate out of state 0 is $\pi_0 p$
  - The rate into the state is $\pi_1 q$
  - Equating the two, we again have:

$$\frac{\pi_0}{\pi_1} = \frac{q}{p}$$
Transition Rates 5

• In general, transition rates across any boundary must be the same
  – For any set of states $A$ in a recurrent chain, we have:

$$\sum_{i \in A} \sum_{j \in A^c} \pi_i p_{ij} = \sum_{i \in A^c} \sum_{j \in A} \pi_i p_{ij}$$
• Example:
  - Assume a connected, undirected graph
  - The only thing known about the graph is that the degree of node \( i \) is \( d_i \)
  - A random walk starts at some initial vertex and moves between the nodes uniformly choosing among the neighbors of each current node

• Is this a Markov chain? What is its matrix \( P \)?
  - Let \( N(i) \) be set of all neighbors of node \( i \)

\[
p_{ij} = \begin{cases} 
  1/d_i & j \in N(i) \\
  0 & \text{otherwise}
\end{cases}
\]
Transition Rates

- Direct solution in Matlab to $\pi = \pi P$ is not possible since it requires the knowledge of $N(i)$ for each $i$
  - Instead, we use the observation that the probability to find the random walk in state $i$ is the combined rate of transitions from all states into $i$

$$\pi_i = \sum_{j=0}^{\infty} \pi_j p_{ji}$$

- Since these terms are non-zero only for neighbors of $i$, we have:

$$\pi_i = \sum_{j \in N(i)} \pi_j p_{ji} = \sum_{j \in N(i)} \frac{\pi_j}{d_j}$$
Transition Rates 8

• Due to normalization by $d_j$, we can guess the shape of the stationary distribution:

$$\pi_i = \frac{d_i}{C}$$

- where $C$ is some constant that we determine below (proving that $\pi$ is unique is beyond our scope)

• We next check if this guess is correct:

$$\pi_i = \sum_{j \in N(i)} \frac{\pi_j}{d_j} = \sum_{j \in N(i)} \frac{1}{C} = \frac{d_i}{C}$$

- and then find out $C$:

$$\sum_k \pi_k = \frac{1}{C} \sum_k d_k = 1 \Rightarrow C = \sum_k d_k$$
Wrap-up

• For computing $E[T]$, use these hints:
  - Pareto:

  $$\int (1-(1-z^{1-\alpha})^k)dz = z \left( 1 - 2F_1\left(\frac{1}{1-\alpha}, -k, \frac{2-\alpha}{1-\alpha}, z^{1-\alpha}\right) \right)$$

  - Exponential:

  $$\frac{1 - z^k}{1 - z} = \sum_{i=0}^{k-1} z^i$$

• Midterm next Thursday
  - Covers everything since the first lecture