Renewal Process Theory II

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Agenda

- Inspection paradox
- Renewal-reward processes
  - Renewal-reward theorem
  - Examples
- Distribution of residuals and age
  - Derivation and examples
- Distribution of spread
  - Examples
Inspection Paradox

• Recall from the previous lecture
  – Age $A(t)$, residual delay $R(t)$, and spread $S(t)$
  – Time $t$ is very large (think of it as uniform between 0 and $\infty$)
**Inspection Paradox 2**

- The “inspection paradox”
  - When we randomly examine a renewal process at time $t$, the average spread $S(t)$ can be much larger than $E[X_j]$

- This can be shown through an example
  - Consider two types of inter-bus delays:

$$P(X_j = 1) = 0.99, P(X_j = 100) = 0.01$$

- This means that 99% of all renewal intervals are 1 minute and the remaining 1% are 100 minutes
  - What is the probability to observe interval $X_j = 100$ at some large time instance $t$?

- Clearly, not 1%
Inspection Paradox 3

- Notice that in the “average sense,” the process has this structure:

- Thus, each 199 minutes (on average) contain one large and 99 small intervals
  - Therefore, the probability that $t$ randomly “lands” into a large interval is $100/199 \approx 50\%$ instead of 1\%
Inspection Paradox 4

- A random observer is more likely to inspect the process during large intervals and experience longer wait times than if he/she arrived at renewal points $Z_n$

- **Definition (pg. 22)**
  - $X$ is *stochastically larger* than $Y$ if
    \[ P(X > t) \geq P(Y > t), \forall t \]
  - For example, uniform $X$ in $[3,5]$ is stochastically larger than uniform $Y$ in $[2,4]$

- **Theorem**: random variable $S(t)$ is stochastically larger than $X_j$
Inspection Paradox 5

• We now have an intuitive explanation of why Pareto wait times in the bus example became larger.
• This is a consequence of very large samples $X_j$ in the Pareto distribution during which we are more likely to inspect the system.
• Example:
  - Compute the expected wait time for the example shown two slides back.
  - Recall:

\[ E[X] = \sum_y E[X|Y = y] P(Y = y) \]
Inspection Paradox 6

• Thus:

\[ E[W] = E[W|small]P(\text{small}) + E[W|\text{large}]P(\text{large}) \]

  where

\[ P(\text{small}) = \frac{99}{199}, \quad P(\text{large}) = \frac{100}{199} \]

• Putting the pieces together:

\[ E[W] = \frac{1}{2} \times \frac{99}{199} + \frac{100}{2} \times \frac{100}{199} = 25.37\ldots \]

• Also note that \( E[X_j] \) is only 1.99 minutes
  – This means that buses arrive at the rate of 30 per hour; however, your expected wait is over 25 minutes!
Renewal Rewards

• Our goal today is to derive the distribution of residual waiting time $R(t)$ and spread $S(t)$
  – Before we do that, we need several results from renewal rewards, which is a segment of renewal process theory

• Suppose each renewal $j$ earns reward $P_j$
  – Sequence $\{P_1, P_2, \ldots\}$ consists of iid random variables

• Then, the cumulative reward in $[0, t]$ is

$$C(t) = \sum_{j=1}^{M(t)} P_j$$

• Technical note
  – Rewards $P_j$ may depend on $X_j$, but not $X_i, i \neq j$
Renewal Rewards 2

• Let the *expected reward* be $c(t) = E[C(t)]$

• The Renewal-Reward Theorem:

$$\lim_{t \to \infty} \frac{C(t)}{t} = \lim_{t \to \infty} \frac{c(t)}{t} = \frac{E[P_j]}{E[X_j]} = \mu E[P_j]$$

• Example:
  
  - You are taking a long test that lasts $t$ time units
  - Suppose it takes $X_j$ seconds to solve problem $j$
  - You earn $P_j$ points for solving the $j$-th problem
  - What is your score at the end of the test?
  - Approximately
Distribution of Residuals

- We are ready to derive the distribution of \( R(t) \)
- Fix \( y \geq 0 \) and define indicator process:

\[
I(t) = \begin{cases} 
1 & R(t) \leq y \\
0 & \text{otherwise}
\end{cases}
\]

- Graphically, it looks like this:
• If we throw a random point $t$ on the interval $[0, T]$, what is the probability to hit a segment with $I(t) = 1$?

$$P(R(t) \leq y) = \frac{1}{T} \int_0^T I(u) \, du$$

• Setting $R$ as the limit of $R(t)$ for $t \to \infty$

$$P(R \leq y) = \lim_{T \to \infty} \frac{1}{T} \int_0^T I(u) \, du$$

• All we now have to do is compute this limit

  - Define reward $P_j$ for renewal $j$ to be:

$$P_j = \int_{Z_{j-1}}^{Z_j} I(u) \, du = \begin{cases} X_j & X_j \leq y \\ y & X_j > y \end{cases}$$

  - Alternatively, $P_j = \min(X_j, y)$
• Next observe that the integral of $I(u)$ is actually the cumulative reward at time $T$:

$$C(T) = \sum_{j=1}^{M(T)} P_j = \int_0^T I(u) du$$

• We neglect special cases when $T$ does not fall on the boundary of $Z_j$ (i.e., partial rewards) as their contribution tends to zero for large $T$

• Next, apply the renewal reward theorem to $C(T)$:

$$\lim_{T \to \infty} \frac{C(T)}{T} = \lim_{T \to \infty} \frac{1}{T} \int_0^T I(u) du = \frac{E[P_j]}{E[X_j]}$$
• Observe that the middle term in the last equation is \( P(R \leq y) \) and re-write:

\[
P(R \leq y) = \frac{E[P_j]}{E[X_j]}
\]

- What is left is to obtain \( E[P_j] \)

• **Lemma**: Suppose \( X \) is a non-negative variable and \( a > 0 \) is some constant. Then

\[
E[\min(X, a)] = \int_0^\infty P(\min(X, a) > x)dx
\]

\[
= \int_0^a P(\min(X, a) > x)dx
\]

\[
= \int_0^a P(X > x)dx
\]

- Reason: for any \( x \leq a \), it follows that \( X > x \) iff \( \min(X, a) > x \)
Distribution of Residuals

• Combining the pieces:

\[ P(R \leq y) = \frac{1}{E[X_j]} \int_0^y P(X_j > x) \, dx \]

• Or in a more digestible form:

\[ P(R \leq y) = \frac{1}{E[X_j]} \int_0^y (1 - F(x)) \, dx \]

• Note: the same technique applies to age:

\[ P(A \leq y) = \frac{1}{E[X_j]} \int_0^y (1 - F(x)) \, dx \]
Distribution of Residuals 6

• Examples:
  - Constant inter-bus delay $s$:
    \[
P(R \leq y) = \begin{cases} 
      y/s & y \leq s \\
      1 & y > s 
    \end{cases}
    \quad \text{uniform}
  
  - Exponential:
    \[
P(R \leq y) = \lambda \int_0^y e^{-\lambda x} \, dx = 1 - e^{-\lambda y}
    \quad \text{exponential}
  
  - Pareto:
    \[
P(R \leq y) = \frac{\alpha - 1}{\beta} \int_0^y (1 + x/\beta)^{-\alpha} \, dx = 1 - (1 + y/\beta)^{1-\alpha}
    \quad \text{Pareto}
\]
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• This explains the observations in homework #1
  – Inter-bus delays with Pareto $\alpha = 3$ had wait times that were Pareto with $\alpha = 2$
  – It also confirms that when the delay between buses is constant, your expected wait is uniform in $[0, s]$

• The final important result to derive is the expected wait time $E[R]$

• Define the residual CDF (also called the equilibrium distribution) by:

$$F_R(y) := P(R \leq y) = \frac{1}{E[X_j]} \int_0^y (1 - F(x)) \, dx$$
• Then the density of wait time is:

\[ f_R(y) = F'_R(y) = \frac{1 - F(y)}{E[X_j]} \]

• Next, we compute the expected wait time:

\[ E[R] = \int_0^\infty y f_R(y) dy = \frac{1}{E[X_j]} \int_0^\infty y (1 - F(y)) dy \]

• To solve the integral, notice:

\[ E[X^2] = \int_0^\infty P(X^2 > x) dx = \int_0^\infty P(X > x^{1/2}) dx \]
Substituting $u = x^{1/2}$ ($dx = 2udu$):

$$E[X^2] = \int_0^\infty 2uP(X > u)du = 2\int_0^\infty u(1-F(u))du$$

Recalling the equation on $E[R]$:

$$E[R] = \frac{1}{E[X_j]} \int_0^\infty y(1-F(y))dy$$

- We have:

$$E[R] = \frac{E[X_j^2]}{2E[X_j]}$$
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• Examples:
  - Constant inter-bus delay:
    \[
    E[R] = \frac{s^2}{2s} = \frac{s}{2}
    \]
  - Exponential:
    \[
    E[X^2] = Var[X] + E^2[X] = 2/\lambda^2 \Rightarrow
    E[R] = \frac{2/\lambda^2}{2/\lambda} = 1/\lambda = E[X_j]
    \]
  - Pareto: derive at home
Distribution of Spread

- Using similar techniques, we can derive the limiting distribution of spread $S(t)$
- The only difference is that we define
  \[
  P_j = \begin{cases} 
  X_j & X_j \leq y \\
  0 & X_j > y 
  \end{cases}
  \]
- The rest of the derivations are similar and lead to:
  \[
  F_S(y) := P(S \leq y) = \frac{1}{E[X_j]} \left( yF(y) - \int_0^y F(x)\,dx \right)
  \]
  - and its density is:
  \[
  f_S(y) = \frac{yf(y)}{E[X_j]}
  \]
Wrap-up

• Example
  – Spread distribution for exponential $X_j$:

$$f_S(y) = \frac{y f(y)}{E[X_j]} = \frac{y \lambda e^{-\lambda y}}{1/\lambda} = \lambda^2 ye^{-\lambda y}$$

• Interestingly, this is the same density as that of a sum of two exponential random variables, i.e., Erlang(2)
  – Why was this intuitively expected?

• For exponential $X_j$, average spread $E[S] = 2E[X_j]$
  – In general,

$$E[S] = E[R] + E[A] = 2E[R] = \frac{E[X_j^2]}{E[X_j]}$$