Renewal Process Theory

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Agenda

• Stochastic processes
  – Definitions
  – Homework #2 examples
• Renewal processes
• More examples
Process Theory

- In many cases, we deal with more than just a single random variable
  - This arises in repeated experiments such as coin tosses, or recurring phenomena such as bus arrivals
- **Definition**: a *process* is an infinite collection of random variables
  - There are two types of processes – discrete and continuous
- A *discrete* process \( \{X(n)\} \) is a *countable* sequence of random variables, where \( X(n) \) is the \( n \)-th variable (often written as \( X_n \))
- If the number of variables is *uncountable*, we have a *continuous* process \( \{X(t)\} \) or \( \{X_t\} \)
Process Theory 2

• The simplest example are Bernoulli coin tosses
  – Sequence $X_1, X_2, \ldots$ is a process
  – Since these are iid (independent identically distributed) random variables, the mean of $X_n$ does not depend on $n$
  – We can thus write $E[X_n] = p$

• It is possible to construct a dependent process
  – Variable $X_n$ depends on previous values $X_1, \ldots, X_{n-1}$
  – For example, autoregressive process $X_n = X_{n-1} + v(n)$, where $v(n)$ are some random variables (often called noise)

• Instance of the $n$-th bus arrival is also a process:
  $$Z_n = \sum_{i=1}^{n} X_i = Z_{n-1} + X_n$$
Another example is the evolution of wealth

- Suppose the more money you have during year \( n - 1 \), the more its increase in year \( n \)
- Write:

\[
X(n) = X(n - 1) + w(n)X(n - 1)
\]

- where \( w(n) \) is some multiplicative random noise (suppose it is uniform in \([0,1] \))

Then what is the distribution of your wealth in year \( n \)?

- Suppose 10,000 people start with \( X(0) = \$1 \); how much skew will there be in year \( n \)?
- Can this model capture the wealth distribution of modern societies?
Clearly, the growth of $X(n)$ is exponential in some “average” sense
- Its mean is computed easily:

\[ E[X(n)] = E[X(0)] \prod_{i=1}^{n} (1 + E[w(i)]) \]

- Assuming that $w(i)$ are iid, we have:

\[ E[X(n)] = E[X(0)](1 + E[w(1)])^n \]

- Average wealth increases multiplicatively from $X(0)$
  - The longer one lives, the more money they are expected to have in this model
What can be said about the distribution of $X(n)$?
- Problem addressed in hw #2

Your goal is to derive the distribution of $X(n)$ and confirm in simulation that your results are accurate

You need to use lognormal random variables
- Variable $X$ is said to be lognormal if there exists some Gaussian random variable $Y$ such that $X = e^Y$
- Excel has its CDF $F(x) = \lognormdist(x, E[Y], \sigma_Y)$

In this case, you can only match the histogram since the CDF function does not exist in closed-form
- A histogram counts the number of samples generated by a random variable in each bin of certain size
• How to compute histogram of the model to compare it to that obtained in simulations?
  - Assume bin \([a_i, a_{i+1})\)
  - Then the model predicts that the fraction of samples in this bin will be \(F(a_{i+1}) - F(a_i)\)

• Alternative methods exist for directly matching the PDF of a distribution, but they are more complex
Renewal Processes

• One important area of process theory deals with renewals (periodic events)
  – Analysis of queues in routers, user arrivals in P2P systems, request backlog in webservers, etc.

• Suppose our system exhibits *recurring* behavior
  – Recurring means that the same situation repeats over time at random time instances
  – Bus arrival is one example from homework #1

• Analysis focuses on queue size distributions, overflow events, and various wait times
  – One common model is an observer that examines the system at some point $t \to \infty$ (asymptotic behavior)
Renewal Processes 2

• Each occurrence of an event is called a renewal
  - Delays between adjacent renewals are given by a sequence of iid random variables $X_1, X_2, \ldots$
  - This forms a discrete process $X_n$

• Here is an illustration:
Renewal Processes 3

- The time (epoch) of the $n$-th renewal (bus arrival):

\[ Z_n = \sum_{j=1}^{n} X_j \]

- Usually $E[X_j]$ is finite and there exists a renewal rate

\[ \mu = \frac{1}{E[X_j]} \]

- It is generally convenient to assume that $X_1$ has a different distribution from $X_n$, $n > 1$
  - Allows construction of stationary processes
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- Let $X_1$ have a CDF $A(x)$ (density $a(x)$) and the remaining $X_j$ have a CDF $F(x)$ (density $f(x)$)

- Then, the distribution of time before the second renewal:

$$P(Z_2 \leq t) = P(X_1 + X_2 < t) = \int_0^t A(t-u)f(u) \, du$$

- Differentiating the above expression, the density of $Z_2$ is given by:

$$[P(Z_2 \leq t)]' = \int_0^t a(t-u)f(u) \, du$$

- which is standard convolution
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• To expand further, denote by $F^{(2)}(t) = F*F$ and by $F^{(n)}(t) = F^{(n-1)}(t)*F(t)$ an $n$-fold convolution
  – This represents the distribution of the sum of $n$ iid variables, each with CDF $F(x)$

• Thus, since $Z_n = X_1 + X_2 + \ldots + X_n$, we get:

$$P(Z_n \leq t) = A(t) * F^{(n-1)}(t)$$

  – which is the distribution of the $n$-th renewal time

• Not a simple metric to compute!
  – But it gets worse
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• Define *renewal process* $M(t)$ to be the number of renewals in the interval $[0, t]$:

$$M(t) = \max\{n \geq 0 : Z_n \leq t\}$$

• $M(t)$ is a continuous-time, discrete-state process

• We can express its tail distribution using $Z_n$:

$$P(M(t) \geq n) = P(Z_n \leq t) = A(t) \ast F^{(n-1)}(t)$$

• This is the same $n$-fold convolution
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- Define renewal function \( m(t) = E[M(t)] \) to be the expected number of renewals in \([0, t]\)

- Recalling the expectation of a non-negative random variable as the sum of the tail

\[
E[X] = \sum_{j=0}^{\infty} P(X > j)
\]

- we get

\[
m(t) = E[M(t)] = \sum_{j=1}^{\infty} A(t) * F^{(j-1)}(t)
\]

- This is not very helpful at all (notice the infinite convolution)
Renewal Processes

• In fact, if we know little about $F(x)$, we cannot compute even the expected number of arrivals in $[0, t]$

• For this reason, much of renewal theory deals with asymptotic results
  – This means that we are interested in systems that have evolved sufficiently long and points $t$ much larger than 0, in which case we have the following result

• The Elementary Renewal Theorem:

\[
\lim_{t \to \infty} \frac{M(t)}{t} = \lim_{t \to \infty} \frac{m(t)}{t} = \mu
\]
Other metrics of interest

- The age $A(t)$ of the process at time $t$ and the residual life $R(t)$ until the next renewal.
• The age is the delay since the last bus left and the residual life is the wait time until the next bus

• We can express both using $M(t)$:

\[ R(t) = \frac{1}{M(t)} + 1 - t \quad A(t) = t - \frac{1}{M(t)} \]

• We can also define spread $S(t)$ to be the current delay between the buses sampled randomly at time $t$

\[ S(t) = A(t) + R(t) = \frac{1}{M(t)} + 1 - \frac{1}{M(t)} \]